

## Glassy behavior of the parking lot model

Amy J. Kolan,\* Edmund R. Nowak,† and Alexei V. Tkachenko‡  
*The James Franck Institute, The University of Chicago, Chicago, Illinois 60637*

(Received 8 June 1998)

We present a theoretical discussion of the reversible parking problem, which appears to be one of the simplest systems exhibiting glassy behavior. The existence of slow relaxation, nontrivial fluctuations, and an annealing effect can all be understood by recognizing that two different time scales are present in the problem. One of these scales corresponds to the fast filling of existing voids, the other is associated with collective processes that overcome partial ergodicity breaking. The results of the theory are in a good agreement with simulation data; they provide a simple qualitative picture for understanding recent granular compaction experiments and other glassy systems. [S1063-651X(99)14402-7]

PACS number(s): 68.45.Da, 61.43.-j, 64.70.Pf

### I. INTRODUCTION

Constrained dynamics of complex systems has long been a subject of extensive experimental and theoretical research. Certain important features, such as slowing kinetics, nonexponential relaxations, and memory effects are believed to be generic for a wide class of systems ranging from structural and spin glasses to granular materials and traffic flows [1–3]. In spite of significant progress in this field there is still no general framework for the description of jamming and glassy phenomena. The development of a clear qualitative picture of them has been frustrated by the relatively high complexity of the considered systems. One could hope to boost the conceptual progress in this field by analyzing simple models capable of capturing the important features of glasses.

In this paper we present a theoretical discussion of one of the simplest systems that exhibits glassylike relaxation dynamics and a nontrivial fluctuation spectrum. This system is known as the parking lot model (PLM) [4–8], or the continuous random adsorption problem, and it is defined as follows. Identical, unit length particles (cars) can adsorb on a line (curb) at rate  $k_+$  per unit curb length. They can also leave the line with rate  $k_-$ . The desorption process is unrestricted while the adsorption is subject to free volume constraints, i.e., two cars cannot overlap (see Fig. 1). This model can be applied in a straightforward way to random physical adsorption of large molecules. In addition, the PLM appears to be one of the most successful models for the description of density relaxation and fluctuations in a vibrated granular material. The possible reason for this is that the dynamics of the PLM drastically depends on the available free volume, just as in the case of granular materials or structural glasses.

The dynamics of the original version of the PLM, in which the particles adsorb irreversibly (i.e.,  $k_- = 0$ ), has been well understood as long ago as 1958 by Renyi. [5] He

found that the system jams at density  $\rho_c \approx 0.75$ , and the way it approaches this state is given by the following formula:

$$\rho(t) = \int_0^{k_+ t} dw \exp \left[ -2 \int_0^w du (1 - e^{-u})/u \right]. \quad (1)$$

The late-stage asymptotics of this result is the power-law relaxation:  $\rho(t) - \rho_c \sim 1/t$ .

The desorption process introduced by Krapivsky and Ben-Naim [8] results in even richer physics. In a recent paper on granular compaction, Nowak *et al.* [9] presented the results of simulations on the PLM. These simulations and the experimental granular compaction data have many important features in common. In particular, the average coverage of the curb as a function of time, shown in Fig. 2, is very similar to the density relaxation curve for the vibrated sand. Once in the steady state, the finite size of the system results in considerable density fluctuations. Insight into their dynamics is provided by the power spectrum as seen in Fig. 3. This figure illuminates one of the most remarkable properties of the parking lot model, i.e., that it exhibits two very different time scales at high  $k \equiv k_+/k_-$  values (high density). These time scales appear in the power spectrum as two corner frequencies, one at high frequency and one at low. The low-frequency corner is Lorentzian, which indicates that it can be associated with exponential relaxation at a single time scale. The high-frequency “corner,” however, shows an unusual “hump” that indicates that the density relaxation can-

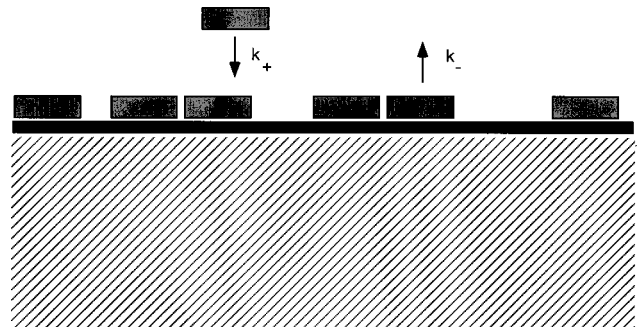


FIG. 1. Parking lot model.

\*Permanent address: Physics Department, St. Olaf College, 1520 St. Olaf Avenue, Northfield, MN 55057.

†Present address: University of Illinois, Loomis Laboratory of Physics, MC-704 1110 West Green Street, Urbana, IL 61801.

‡Author to whom correspondence should be addressed.

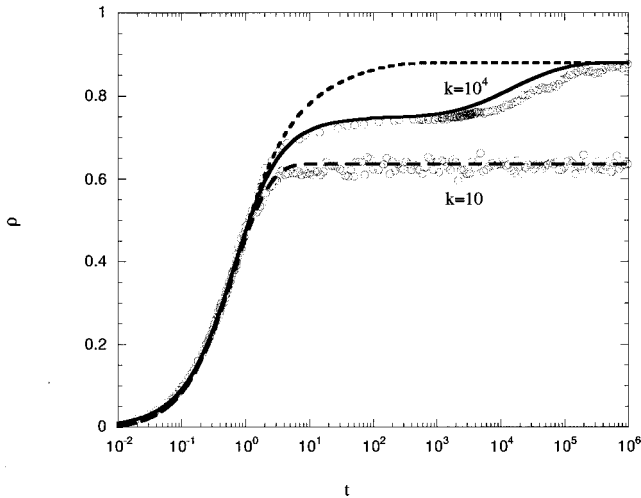


FIG. 2. Simulation data (circles) and theoretical results for density evolution in the cold ( $k=10^4$ ) and hot ( $k=10$ ) regimes. Note that the mean field (dashed line) is quite adequate for the description of the hot system but it fails to describe the slow dynamics at the “supercooled” regime for  $k=10^4$ . The solid line is the combination of the irreversible parking curve, Eq. (1), describing the fast stage of the relaxation (for  $k=10^4$ ), and our result, beginning at  $t=100$ , for the later slow dynamics.

not be described with only one time scale. It is these features in the power spectrum that led Nowak *et al.* to present the parking lot model in conjunction with the sand experiment, which shows a similar nontrivial fluctuation spectrum. The existence of several relaxation time scales is a signature of *the partial ergodicity breaking* exhibited by the model, i.e., its high-frequency evolution does not allow the system to explore all the configurational space. In this sense the observed behavior of the PLM may be relevant not only for understanding the particular granular compaction experiment but also for the whole class of systems exhibiting glassy relaxation dynamics. Below we focus on developing an analytical description of the PLM capable of capturing these intriguing features and revealing the underlying physics.

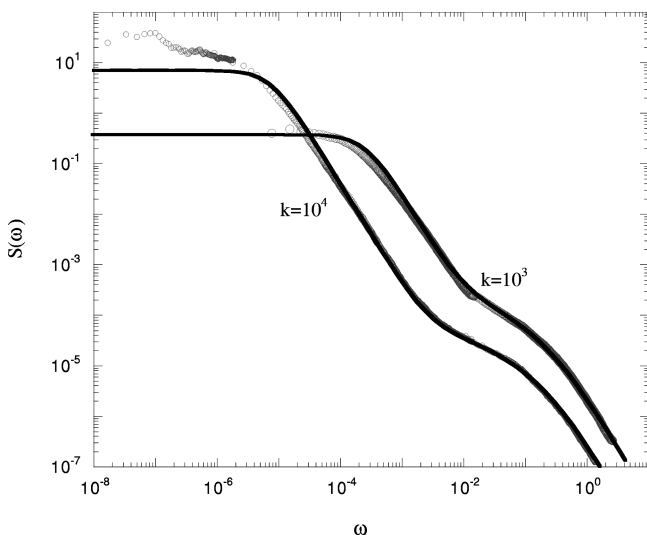


FIG. 3. The power spectrum of density fluctuations near the equilibrium for  $k=10^3$  and  $k=10^4$ . Circles and solid lines represent the simulation data and our analytic results, respectively.

## II. LIMITATIONS OF THE MEAN-FIELD APPROACH

In their original work on the reversible PLM [8], Krapivsky and Ben-Naim proposed a mean-field description of the problem that can be essentially expressed in terms of the following master equation for the average density  $\rho$ :

$$\frac{\partial \rho}{\partial t} = k_+(1-\rho)\exp\left(-\frac{\rho}{1-\rho}\right) - k_-\rho. \quad (2)$$

Here the first term represents the adsorption rate and takes into account its strong dependence on the available free volume; the second term corresponds to unrestricted desorption. The above equation indeed captures some of the features of the PLM. In particular, it results in an equation for the equilibrium density  $\rho_{eq}$ , which is consistent with the numerical data:

$$\left(\frac{1-\rho_{eq}}{\rho_{eq}}\right)\exp\left(-\frac{\rho_{eq}}{1-\rho_{eq}}\right) = \frac{k_-}{k_+}. \quad (3)$$

Although this mean-field approach does result in typical jamming dynamics, the predicted density relaxation rate is several orders of magnitude faster than that observed in the simulations [9]. Moreover, the mean-field description fails to capture the most interesting feature of the PLM, its fluctuation spectrum. Indeed, the very structure of Eq. (2) cannot result in anything different from a regular linear response equation for the density near equilibrium, i.e., it should have a single relaxation time for small fluctuations.

An essential feature that the above mean-field approach overlooks is the strong correlation between adsorption and desorption events. This correlation is familiar to anyone who has ever attempted to park in a big city. Cars do not leave the curb very often but as soon as they do an incoming car rushes in to take the newly created space. This paired adsorption-desorption process does not change the density of the system; it is equivalent to merely sliding a car in its parking space. One can explicitly emphasize this by the introduction of a slow (adiabatic) variable  $Z \equiv N + N^*$ , which is the sum of the number of adsorbed cars,  $N$ , and the number of voids large enough to fit at least one particle  $N^*$ . If the system is sufficiently dense, most individual adsorption or desorption events do not change the parameter  $Z$ . The existence of this slow variable results in the observed separation of the relaxation time scales. The fast modes of the density relaxation correspond to the evolution with nearly constant  $Z$ , while the slow dynamics is determined by low-probability events, which result in a drift of  $Z$ . In other words, on short time scales the ergodicity of the PLM is broken: the system can explore only the part of the configurational space corresponding to a constant  $Z$ .

The fast and slow modes are expressed as two typical time scales in the density fluctuations and also lead to a two-stage relaxation process of the density evolution from  $\rho=0$  to  $\rho_{eq}$ . In the case of a very small desorption rate  $k_-$  the system follows the universal irreversible-adsorption curve, Eq. (1), until it jams at  $\rho_c \approx 0.75$ . Afterward, it slowly evolves towards the equilibrium density. By using the above result for  $\rho_{eq}$  as a function of  $k \equiv k_+/k_-$  and knowing  $\rho_c$  one can construct the “kinetic phase diagram” of the system (see Fig. 4): if  $\rho_{eq}(k) < \rho_c$  the system reaches the equilibrium

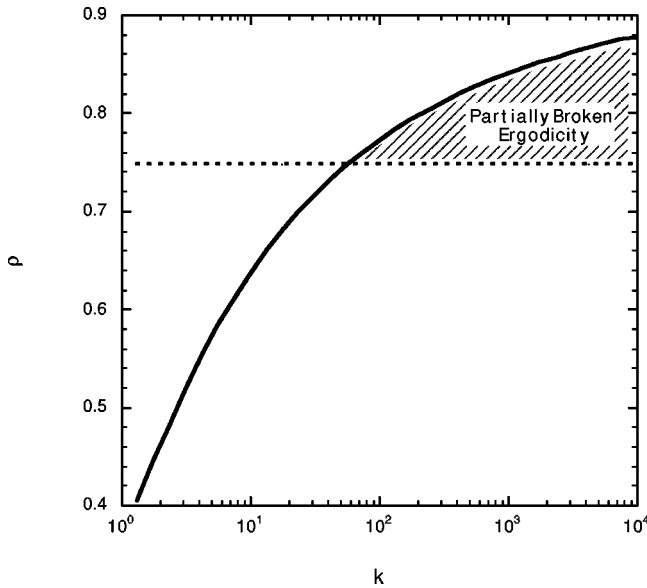


FIG. 4. “Kinetic phase diagram” of the system. The solid line corresponds to the reversible equilibrium line  $\rho_{eq}(k)$ ; the dashed line shows the jamming density above which the ergodicity is partially broken.

density before it jams, otherwise it relaxes to  $\rho_c$  and then slowly creeps toward equilibrium. It would be tempting to associate the critical point  $k_c \approx 60$ , which separates the two regimes, with a glass transition. However, this term is traditionally reserved for the hypothetical point at which the kinetic coefficients of a system would go to zero (in our simple case this is the point  $k = \infty$ ). Thus, the zone of partial ergodicity breaking appearing at  $k > k_c$  is an analog of what is conventionally called a supercooled liquid rather than a glass.

An important implication of the above picture of the PLM kinetics is the existence of an annealing effect, which is typical for classical glassy systems and has also been observed in the granular compaction experiments [9]. The idea is that one can overcome the slow kinetics of the supercooled system by “heating” it (by decreasing  $k$  below  $k_c$ ), then slowly cooling it (increasing  $k$ ), so that the system would follow the reversible equilibrium curve  $\rho_{eq}(k)$ .

### III. FAST DYNAMICS OF THE SYSTEM

As we have pointed out above, the fast dynamics of the PLM is dominated by a two part process: a desorption that takes place at rate  $k_-$  leaving a void and an adsorption that occurs at rate  $k_+ z$ , where  $z$  is the void size minus a car length. This two-part process is equivalent to simple replacement of one adsorbed particle with another or a sliding of a single particle in its own space. As a result, the separations between a given particle and its two nearest neighbors,  $x_1$  and  $x_2$ , change randomly to the new values  $x'_1$  and  $x'_2$  in such a way that their sum remains constant:  $x_1 + x_2 = x'_1 + x'_2 = z$ . This sliding process creates the necessary mixing to ensure that the steady-state distribution of the spaces between cars,  $p(x)$ , corresponds to the maximal entropy for a fixed sum of all the separations. Thus, the distribution function can be obtained by maximizing the functional

$$\Phi[p] = - \int_0^\infty dx \left( p(x) \ln p(x) + p(x) \frac{x}{\delta} \right) \quad (4)$$

with respect to  $p(x)$ , where  $1/\delta$  is the Lagrange multiplier conjugate to  $x$ . Naturally, the resulting formula for  $p(x)$  is similar to the Boltzmann distribution:

$$p(x) = \frac{1}{\delta} \exp\left(-\frac{x}{\delta}\right). \quad (5)$$

The parameter  $\delta$  is equal to the average spacing between cars, i.e.,

$$\delta = \frac{1 - \rho}{\rho}. \quad (6)$$

We now proceed with the calculation of the high-frequency part of the fluctuation spectrum. The coupled adsorption-desorption process can be viewed as a relaxation-excitation of the conventional two-state (telegraph) system, which can switch from one state, 0, to another one, 1, with a characteristic time  $t$  and then relax back to 0 with the characteristic time  $\tau$ . The power spectrum of telegraph noise has been investigated by Machlup [10] and is given by

$$S(\omega) = \frac{1}{\pi(t + \tau)} \frac{1}{\omega^2 + \nu^2}, \quad (7)$$

where  $\nu = 1/\tau + 1/t$ .

The fast dynamics of the density is the superposition of these nearly independent telegraph modes, whose number is essentially the number of cars (or, more precisely,  $Z$ ). An important feature of the adsorption-desorption modes is that while their “excitation rate,”  $1/t = k_-$ , is uniform over the system, the relaxation time is the property of an individual excited state, i.e., the adsorption rate is proportional to the free volume available at a given void:  $1/\tau = k_+(x_1 + x_2)$ . As a result, the fluctuation spectrum deviates from the simple Lorentzian form, Eq. (7):

$$S_H(\omega) \equiv \langle \rho_\omega \rho_{-\omega} \rangle \equiv \frac{\langle N_\omega N_{-\omega} \rangle}{L^2} = \frac{\rho k_-}{\pi L} \int_0^\infty \frac{f(\nu) d\nu}{\omega^2 + \nu^2}. \quad (8)$$

Here  $L$  is the total length of the system. Note that here and below we assume that  $k \equiv k_+/k_- \gg 1$ , so that the typical adsorption process is much faster than the desorption. As a result, the desorption rate determines the amplitude in the above expression, while the (nonuniform) adsorption dominates its frequency dependence [i.e.,  $\nu \approx 1/\tau = k_+(x_1 + x_2)$ ]. Thus, the calculation of the power spectrum for the parking lot model reduces to finding the distribution of relaxation times for adsorption, and then weighting the spectrum for telegraph noise with this distribution. As shown above, the distribution function for the inter-car spaces is exponential. Assuming that the separations between a given car and its two neighbors,  $x_1$  and  $x_2$ , are uncorrelated, we obtain the following distribution of relaxation rates,  $\nu = k_+(x_1 + x_2)$ :

$$f(\nu) = \frac{\nu}{\omega_H^2} \exp\left(-\frac{\nu}{\omega_H}\right). \quad (9)$$

Here

$$\omega_H \equiv k_+ \delta \quad (10)$$

is the characteristic frequency of the fast relaxation.

One can now substitute the distribution function for  $\nu$  into the above expression for the high-frequency fluctuation spectrum, Eq. (8). The result of the numerical integration of this formula over  $\nu$  is in excellent agreement with the simulations, see Fig. 3. We now have an explanation for the hump in the spectrum at high frequency; it is caused by a relatively broad distribution of relaxation times for the adsorption process. Further examination of our result illuminates the connection between the PLM and other, more complicated systems exhibiting glassy dynamics. The above distribution of relaxation rates does not just distort the Lorentzian form near the characteristic frequency,  $\omega_H \equiv k_+ \delta$ , but rather affects the spectrum at all frequencies below  $\omega_H$ . The asymptotic behavior of the spectrum at  $\omega \ll \omega_H$  is

$$S(\omega) \sim \ln \omega. \quad (11)$$

Hence, the spectrum never recovers the Lorentzian-like plateau regime at low frequencies. The logarithmic behavior is reminiscent of the power-law spectrum typical for glasses at mesoscopic frequencies (it corresponds to  $\beta$  relaxation [1]). Because the logarithmic behavior is a result of the problem being one dimensional, one could expect real power-law behavior in the spectrum in higher dimensions. A simple assumption that the adsorption probability is proportional to the free volume associated with the newly created void, combined with the natural exponential distribution for the free volume, would result in the  $1/\omega$  spectrum at mesoscopic frequencies:

$$S_H(\omega) \sim \int_0^\infty \frac{\exp(-\nu/\nu_0) d\nu}{\omega^2 + \nu^2} \sim 1/\omega, \quad \omega \ll \nu_0. \quad (12)$$

Note that in the one-dimensional case discussed above, the free volume is a sum of two presumably independent variables,  $x_1$  and  $x_2$ , each of which has an exponential distribution. As a result, the distribution function for the free volume vanishes near zero, i.e., the probability of finding a long-living excited state is strongly suppressed in the one-dimensional case compared to the higher dimensions.

#### IV. SLOW DYNAMICS OF THE SYSTEM

In the previous section we have discussed the fluctuations of the density on short time scales, over which ergodicity is effectively broken. Now we proceed to a discussion of the low-frequency part of the fluctuation spectrum associated with the change of the slow parameter  $Z$ . This change is caused by collective events; the rearrangement of a state corresponding to a given  $Z$  is dominated by a two-car process. In one process, responsible for decreasing the ‘‘ground state’’  $Z$  by 1, two adjacent cars leave and a single one (a ‘‘bad’’ parker) comes in their stead, hogging the space; the opposite process results in adding an extra car to the lot: a car exits and leaves a large space big enough for two cars, provided that the new cars are ‘‘good’’ parkers. How do these collective modes affect the power spectrum? We now

calculate the rates of the above two-car processes. The rate of the  $-1$  process has three contributions. First, a car must leave, and the corresponding ‘‘trial rate’’ is just  $k_-$  per particle. Then, an adjacent car must leave before the hole left by the first car fills, which gives a ‘‘waiting’’ factor  $2k_- \int_0^\infty \int_0^\infty \exp(-\nu t) dt f(\nu) d\nu$ . Finally, the big hole must be blocked by a bad parker (up to a correction of order of  $\delta$ , the probability of this is unity). Thus the overall rate is

$$\nu_{-1} = \frac{2k_-^2}{k_+ \delta}. \quad (13)$$

The opposite,  $+1$ , process has the same trial frequency,  $k_-$  per car. The void left by the car (its length is  $x_1 + x_2 + 1$ ) must be large enough for two cars, which gives the factor

$$\frac{1+\Delta}{\delta^2} \exp\left[-\frac{1+\Delta}{\delta}\right] d\Delta. \quad (14)$$

Note that the first incoming car must park with precision  $\Delta = x_1 + x_2 - 1$  in order to leave enough space for the second car. The probability of this happening contributes a factor of  $2\Delta/(1+\Delta)$  to the overall  $+1$  rate:

$$\nu_{+1} = k_- \int_0^\infty \frac{2\Delta}{\delta^2} \exp\left[-\frac{1+\Delta}{\delta}\right] d\Delta = 2k_- \exp(-1/\delta). \quad (15)$$

The density changes with time according to the following equation:

$$\dot{\rho} = (\nu_{+1} - \nu_{-1})\rho + \eta(t). \quad (16)$$

Here  $\eta$  is the noise originating from the fact that the density changes by discrete one-particle steps.  $\langle \eta(t) \rangle = 0$ , because the average evolution is given by the interplay of the  $+1$  and  $-1$  kinetic terms in the above equation. Since adding an extra car or removing one at the moment of time  $t_0$  corresponds to  $\dot{N} = \pm \delta(t - t_0)$ , and since there is no obvious mechanism for the correlations between such processes,

$$\langle \eta(t) \eta(t') \rangle = \frac{\nu_{-1} + \nu_{+1}}{L} \rho \delta(t - t'). \quad (17)$$

Using Eq. (16), we find that the equilibrium density is determined by the condition

$$\frac{k_-}{k_+} = \delta_{eq} \exp(-1/\delta_{eq}), \quad (18)$$

where  $\delta_{eq} = \rho_{eq}^{-1} - 1$ . This result coincides with the mean-field one, Eq. (3). We note that the mean-field approach ignores the adsorption-desorption correlations; this would be a reasonable assumption for the model with strong diffusion of the adsorbed cars. Since the diffusion cannot shift the equilibrium properties, it is not surprising that the mean-field approach gives the correct value of  $\rho_{eq}$ . As to the description of the PLM kinetics, the rates  $\nu_{-1}$  and  $\nu_{+1}$  in Eq. (16) differ by an exponentially small factor,  $2 \exp(-1/\delta_{eq})$  from their mean-field analogs. The above equation describes only

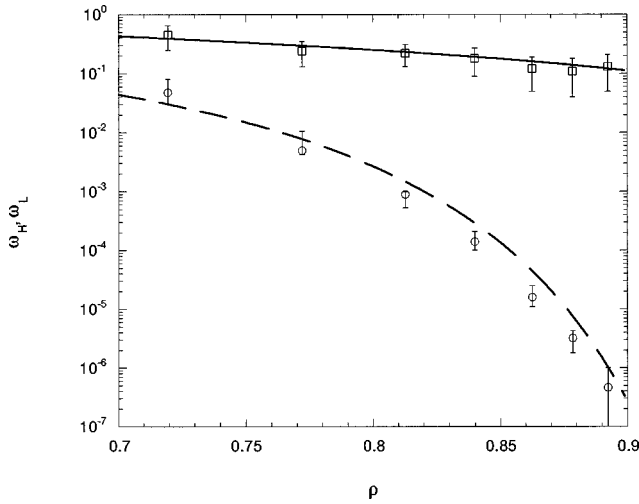


FIG. 5. Low and high characteristic frequencies,  $\omega_L$  and  $\omega_H$ , as functions of the equilibrium density. The dashed and solid lines represent the theoretical results; the squares and the circles are the simulation data for  $\omega_L$  and  $\omega_H$ , respectively.

the slow evolution of the jammed state toward equilibrium. Thus, the overall density relaxation curve of the “cold” system (at  $k > k_c$ ) consists of the classical fast regime, Eq. (1), resulting in a jamming at  $\rho = \rho_c \approx 0.75$ , and the desorption-promoted final stage, discussed here. Such a combination of the two theoretical results is in agreement with the simulation data, as is shown in Fig. 2. The same figure shows that the mean-field curve does not capture the two-stage nature of the relaxation dynamics and is inadequate for the description of the cold system ( $k = 10^4$ ). However, the mean field may be used for the description of the single-stage relaxation of the “hot” system.

By expanding Eq. (16) near the equilibrium density, one can determine the relaxation frequency of the system,  $\omega_L$  and the spectrum of the low-frequency fluctuations:

$$\omega_L = \frac{2k_+}{\delta_{eq}} \exp(-2/\delta_{eq}), \quad (19)$$

$$S_L(\omega) = \frac{2k_- \rho_{eq}}{\pi L} \frac{\exp(-1/\delta_{eq})}{(\omega^2 + \omega_L^2)}. \quad (20)$$

By combining this expression with the earlier result for the high-frequency fluctuations, one obtains the following analytic form for the entire power spectrum of the PLM:

$$S(\omega) \equiv S_H(\omega) + S_L(\omega) = \frac{k_- \rho_{eq}}{\pi L} \left( \int_0^\infty \frac{\exp(-\nu/\omega_H)}{\omega^2 + \nu^2} \frac{\nu d\nu}{\omega_H^2} + \frac{2 \exp(-1/\delta_{eq})}{\omega^2 + \omega_L^2} \right). \quad (21)$$

This result agrees amazingly well with the simulation data as shown in Fig. 3. Figure 5 shows how the characteristic frequencies  $\omega_H$  and  $\omega_L$  depend on the control parameter  $k$

$\equiv k_+/k_-$  (or, equivalently on  $\rho_{eq}$ ); the theoretical calculation is again in good agreement with simulation.

## V. CONCLUSIONS

We have presented a theoretical discussion of the parking lot model, which appears to be a very simple glassy system, perhaps the simplest. We have identified two time scales in the problem: one associated with a simple relaxation of voids and the other corresponding to the collective (two-particle) processes responsible for the rearrangement of the “ground state” (the state that the system can reach by an instant filling of all currently available voids). In the limit of weak desorption, corresponding to a large difference between the two times, the relaxation of the system toward its equilibrium density is a two-stage process: first, it reaches the universal jamming density  $\rho = \rho_c \approx 0.75$  as if there were no desorption at all, then it slowly relaxes to  $\rho_{eq}(k)$  via collective rearrangements. This two-stage relaxation feature disappears in the regime of strong desorption ( $k < k_c$ ), and we identify the crossover point  $k_c$  with  $\rho_{eq}(k) = \rho_c$ .

At  $k > k_c$ , on times shorter than the longest characteristic scale ( $\omega_L^{-1}$ ) the evolution of the system is nonergodic; this regime is analogous to a supercooled liquid. The system evolves by jumping between the metastable states corresponding to different values of the parameter  $Z$ . Note that the lifetime of these states does not grow with the system size, but rather decreases. Since the probability of the rearrangement of the ground state is  $\nu_{+1} + \nu_{-1}$  per particle per unit time, its lifetime is inversely proportional to the number of cars  $N \approx Z$ :  $\tau_Z^{-1} = Z(\nu_{+1} + \nu_{-1})$ . In this sense, the free energy landscape of PLM is similar to that of structural glasses. Another similarity between the PLM and glassy systems is the possibility of accelerating relaxation by means of annealing.

The existence of two characteristic time scales is responsible for the intriguing form of the fluctuation spectrum of the reversible parking problem. The slow fluctuations are described by a single Lorentzian with the relaxation frequency  $\omega_L$  associated with two-particle rearrangements. The fast dynamics is a superposition of many single-particle adsorption-desorption modes. In a sense, the non-Lorentzian form of the high-frequency part of the power spectrum is a reflection of the deviations of the local density from its average value, i.e., it is a signature of long-living disorder. In this form, our observation may be relevant for understanding the nontrivial behavior of the relaxation spectrum of other glassy systems on mesoscopic frequencies. The distribution of the relaxation rates of the single-particle excitations results not only in the distortion of the Lorentzian in the vicinity of the characteristic corner frequency  $\omega_H$ , but also in an interesting logarithmic behavior of the power spectrum at lower frequencies ( $S \sim \ln \omega$ ). We have suggested that this feature is reminiscent of the power-law spectrum corresponding to  $\beta$  relaxation in classical glasses and that such a power-law behavior could be reproducible in PLM at higher dimensions.

## ACKNOWLEDGMENTS

We thank H. M. Jaeger, S. R. Nagel, L. P. Kadanoff, S. Coppersmith, T. A. Witten, J. Cederberg, B. Cipra, and M.

Richey for inspiring and valuable discussions. The work was supported in part through the MRSEC program by the National Science Foundation through Grant Nos. NSF DMR 9400379, NSF DMR 9415605, and NSF DMR 9528957. We

express our gratitude to the Pew Midstates Science and Mathematics Consortium for funds provided through their short term consultation program, and to St. Olaf College for support through its sabbatical leave program.

- 
- [1] *Non-Debye Relaxation in Condensed Matter*, edited by T. V. Ramakishnan and M. Raj Lakshmi (World Scientific, Singapore, 1987).
- [2] M. Mezard, G. Parizi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
- [3] *Traffic and Granular Flow*, edited by D. E. Wolf, M. Shreckenberg, and A. Bachem (World Scientific, Singapore, 1996).
- [4] J. V. Evans, *Rev. Mod. Phys.* **65**, 1281 (1993).
- [5] A. Renyi, *Publ. Math. Inst. Hung. Acad. Sci.* **3**, 109 (1958).
- [6] X. Jin, G. Tarjus, and J. Talbot, *J. Phys. A* **27**, L195 (1994).
- [7] V. Privman and M. Barma, *J. Chem. Phys.* **97**, 6714 (1992).
- [8] P. L. Krapivsky and E. Ben-Naim, *J. Chem. Phys.* **100**, 6778 (1994).
- [9] E. R. Nowak, J. B. Knight, E. Ben-Naim, H. M. Jaeger, and S. R. Nagel, *Phys. Rev. E* **57**, 1971 (1998).
- [10] S. Machlup, *J. Appl. Phys.* **25**, 341 (1954).